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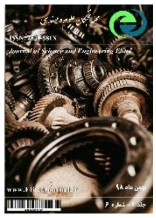

Graphs of Lattice Implication Algebras Based on LI- ideal

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Abstract

In this paper, two types of graphs of lattice implication algebras will be studied. To do so, the notion of equivalence relation $\equiv_A$ of lattice implication algebra is first introduced. Then two types of graphs $\Gamma^A(L)$ and $\Gamma_A(L)$ are defined, respectively. Moreover, we investigate basic properties of these graphs such as planarity, connectivity, regularity, chordality, among others.

Keywords: Lattice implication algebra, Chordal graph, Clique, Planar graph, Girth, LI-ideal.

1- Introduction

In order to investigate a many-valued logical system whose propositional value is given in a lattice, in 1993 Y. Xu [14]first established the lattice implication algebra by combining lattice and implication algebra, and explored many useful structures. The ideal theory serves a vital function for the development of lattice implication algebras. Y. Xu, Y. B. Jun and E. H. Roh [8] introduced the notion of LI-ideals of lattice implication algebras. In particular, Y. B. Jun [7] introduced the concept of prime LI-ideals of lattice implication algebras and discussed some of their properties. Ke Yun Qin, Y. Xu and Y. B. Jun [11] introduced the notion of ultra LI-ideal in lattice implication algebras. Making connection between various algebraic structures and graph theory by assigning graphs to an algebraic structures and investigating the properties of one from the another is an exciting research methods in the last decade. For example, Beck in [4] associated a graph to a ring R is the zero-divisor graph. It is a simple graph with vertex set $Z(R) - \{0\}$, two vertices x and y are adjacent if and only if $xy = 0$. Furthermore, Barati et al.[2] associated a simple graph $\Gamma_S(R)$ to a multiplicatively closed subset S of a commutative ring R with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in S$. Afkhami et al. [1] introduced the same graph structure on a lattice. They considered a lattice L and defined a graph $\Gamma_S(L)$ with all elements of L as vertices and two distinct vertices $x, y \in L$ are adjacent if and only if $x \vee y \in S$, where S is a subset of L which is closed under $\wedge$ operation. Also, Jun and Lee in [9] defined the concept of associated graph of BCI/BCK-algebra and verified some properties of this graph. Zahiri and Borzooei in [17] associated a new graph to a BCI- algebra X represented by $G(X)$, where this definition is based on branches of X. Tahmasbpour in [12, 13] studied chordality of the graph defined by Borzooei, Zahiri and introduced four new graphs of BCI/BCK-algebras constructed by equivalence classes determined by ideal I and dual ideal $I^{\vee}$. Furthermore, in [14, 15] introduced two new graphs of BCK- algebras based on fuzzy ideal μ_I and fuzzy dual ideal $\mu_{I^{\vee}}$, two new graphs of lattice implication algebras based on fuzzy filter μ_F and fuzzy LI- ideal μ_A. This paper is divided into four parts. In Section 2, we recall some concepts of graph theory such as connected, planar, outerplanar, Eulerian, chromatic number, clique number, among others. Section 3, is an introduction to general theory of lattice implication algebras. We will first give the notions of lattice implication algebra, and investigate their elementary and fundamental properties and deal with a number of basic concepts, such as congruence, LI-ideal, among others. In Section 4, we introduce the associated graphs $\Gamma^{A}(L)$ and $\Gamma_{A}(L)$.

2- Preliminaries of graph theory

In this section, we put together some well-known concepts, most of which can be found in [4]. We begin by recalling some of the basic terminologies from the theory of graphs. Needless to mention that all graphs considered here are simple graphs, that is, without loops or multiple edges.

Definition 2.1.([4])A graph $G = (V, E)$ is connected if any of vertices x and y of G are connected by path in G; otherwise, the graph is disconnected. A graph G is called a complete graph on n vertices if $|V(G)| = n, xy \in E(G)$, for any distinct element $x, y \in V(G)$, denoted by K_n. For any $T \subseteq V(G)$, the graph with vertex set $V(G) - T$ and edge set $E(G) - T'$ is denoted by $G - T$, where $T' = \{xy \in E(G); x \in T, y \in G\}$. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is called a star graph if there exists a vertex x in G such that every vertex in G connected to x and other vertices in G do not connect to each other. In graph G with vertex set $V(G)$, the distance between two distinct vertices x and y, denoted by $d(x, y)$, is the length of the shortest path connecting x and y, if such a path exists; otherwise, we set $d(x, y) := \infty$. The diameter of a graph G is $diam(G) := \sup\{d(x, y); x, y \in V(G)\}$. Also, the girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle in G if G has a cycle; otherwise, we get $gr(G) := \infty$. For a vertex x in graph G, the neighborhood of x is the set of vertices adjacent to x, denoted by $N_G(x)$ and $N_G[x] = N_G(x) \cup \{x\}$ and $\deg(x) = |N_G(x)|$. A graph G is called regular of degree k when every vertex has precisely k neighbors. A cubic graph is a graph in which all vertices have degree three. In other words, a cubic graph is a 3-regular graph. A graph G is chordal if every cycle of length at least 4 has a chord, which is not part of the cycle but connects two vertices of the cycle. The greatest induced complete subgraph denotes a clique . If graph G contain a clique with n elements, and every clique has at most n elements, we say that the clique number of G is n and write $\omega(G) = n$. A graph G is called $k-$partite when its vertex set can be partitioned into $k-$disjoint parts $X_1, X_2, \dots, X_k$, so that for $x, y \in X_i, i = 1, \dots, k$, we have $xy \notin E(G)$, for $x \in X_i, y \in X_j, i \neq j, i, j = 1, \dots, k$, we have $xy \in E(G)$. The complete bipartite (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. Moreover, for distinct vertices x and y, we use the notation $x - y$ to show that is x connected to y. A subset A of the vertices is called an independent set if the induced subgraph on A has no edges. The maximum size of an independent set in a graph G is called the independence number of G and is denoted by $\alpha(G)$. Let $P = (V, \leq)$ be a poset. If $x \leq y$ but $x \neq y$, then we write $x < y$. If x and y are in V, the y covers x in P if $x < y$ and there is no z in V with $x < z < y$. Two sets $\{x \in P; x\ covers\ 0\}$ and $\{x \in P; 1\ covers\ x\}$ denoted by $atom(P)$ and $coatom(P)$, respectively. Let $L \subseteq P$, we say L is a chain if for all $x, y \in L, x \leq y$ or $y \leq x$. Chain L is maximal if for all chain $L', L \subseteq L'$ implies that $L = L'$. Two vertices of G are orthogonal, denoted by $x \perp y$, if x and y are adjacent in G and there is no vertex $z \in G$, which can be adjacent to both x and y. A graph G is called complemented if for each vertex x of G, there is a vertex y of G, such that $x \perp y$. A set S is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The dominating number $\gamma(G)$ is the minimum cardinality of a dominating set in G.

Definition 2.2. The lower, upper neighbors of an arbitrary element x in L are the sets $B^l(x) := \{y \in L; x\ covers\ y\}$ and $B^u(x) := \{y \in L; y\ covers\ x\}$, respectively. Also, for every subset A and B of L we put $L_A^B := \{B\}^l - \{A\}^l$ and $U_A^B := \{B\}^u - \{A\}^u$.

Definition 2.3.([4]) A walk or path graph has vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$ such that edge e_k joins vertices v_k and v_{k+1} , denoted by P_n. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Graph G is planar if it can be drawn on the plane without edges having to cross. Proving that a graph is planar amounts to redrawing the edges such that no edges will cross. The vertices may have to be moved around and the edges drawn in an indirect manner. Kuratowski's theorem says that a finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. The chromatic number of any planar graph is less than or equal to 4.

Example 2.4. Figure 1 shows that 3-cube and complete graph K_4 are planar.

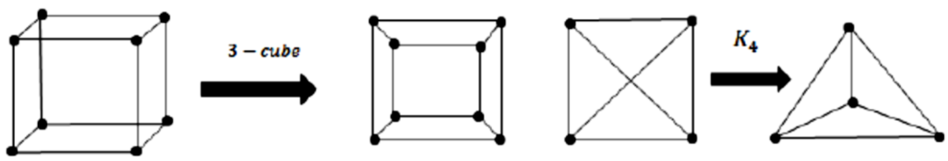

Figure 1

Definition 2.5.([4]) Let G be a plane graph. The connected pieces of the plane which remain when the vertices and edges of G are removed are called the region of G. A faces marks a region bounded by edges . An undirect graph is an outerplanar graph if it can be drawn in the plane without crossing such that all of the vertices belong to the unbounded face of drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Definition 2.6.([10]) Number g is called the genus of the surface if it is homeomorphic to a sphere with g handles or equivalently holes. Besides, genus g of a graph G is meant to be the smallest genus of all surfaces so that graph G can be drawn on it without edge-crossing. The graphs of genus 0 are precisely the planar graphs since the genus of plane is zero. The notation $\gamma(G)$ stands for the genus of a graph G.

Theorem 2.7. ([10]) For positive integers m and n, we have:

(i) $\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$, if $n \geq 3$.

(ii) $\gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$, if $m, n \geq 2$, where $\lceil x \rceil = \min\{ n \in Z | x \leq n\}$.

Definition 2.8. ([3]) Let G be a graph with n vertices that are assumed to be ordered from v_1 to v_n. An $n \times n$ matrix A, in which $a_{ij} = 1$ if there exists an edge from v_i to v_j, $a_{ij} = 0$; otherwise, is a adjacent matrix of G. The characteristic polynomial of matrix G is also denoted by $\chi(G, \lambda)$, which is $\det(\lambda I - G)$.

Theorem 2.9. ([3]) Let G be the complete graph K_n with n vertices. Therefore, $\chi(G, \lambda) = (\lambda - n + 1)(\lambda + 1)^n$.

Definition 2.10. ([4]) An Eulerian path is a path in a finite graph that visits every edge exactly once. Similarly, an Eulerian cycle is an Eulerian path which starts and ends on the same vertex. Euler's theorem says that a connected graph G is Eulerian if and only if all vertices of G are of even degrees.

Definition 2.11. ([6]) A lattice is an algebra $L = (L, \wedge, \vee)$ that satisfies the following conditions, for all $a, b, c \in L$.

(i) $a \wedge a = a, a \vee a = a$.
(ii) $a \wedge b = b \wedge a, a \vee b = b \vee a$.
(iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c), a \vee (b \vee c) = (a \vee b) \vee c$.
(iv) $a \vee (a \wedge b) = a \wedge (a \vee b)$.
Definition 2.12. ([6]) Let $P = (X, \leq_P)$ be a poset. Therefore, a comparability graph (com-graph) of $P = (X, \leq_P)$ is the graph $Com(P) = (X, E_{com(P)})$, where $xy \in E_{com(P)}$ if and only if x is comparable with y in P.
Definition 2.13. [6] A poset or lattice is (upper) semi-modular if, whenover two elements have a common lower cover, they have a common upper cover; (lower) semi-modularity is defined dually. Let $G = (V, E)$ be a graph with a vertex set V and an edge set $E \subseteq V \times V$. It is n −connected if the restriction of G to the vertices $V - C$ is connected, whenever $C \subseteq V$ has fewer than n elements. A chain C has rank d if its cardinality of C is $d + 1$. A poset is ranked at d if every maximal chain has the rank d.
Theorem 2.14. ([6]) Let L be a (finite or infinite) semi-modular lattice of rank d that is not a chain. Therefore, the comparability graph of L is $(d + 1)$ −connected if and only if L has no simplicial element, where $z \in L$ is simplicial if the elements comparable to z form a chain.

3- Preliminaries of lattice implication algebras

In this section, we introduce the concepts of ultra, obstinate, prime, maximal LI-ideal, the relationships between these LI-ideals are described.
Definition 3.1. ([16]) Let $(L, \vee, \wedge, 0, I)$ be a bounded lattice with an order reversing involution $'$, I and 0 the greatest and smallest elements of L, respectively. Then $(L, \vee, \wedge, ', \rightarrow, 0, I)$ is called a lattice implication algebra if the following conditions hold for any $x, y, z \in L$:
(i) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(ii) $x \rightarrow x = I$;
(iii) $x \rightarrow y = y' \rightarrow x'$;
(iv) $x \rightarrow y = y \rightarrow x = I$ implies $x = y$;
(v) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;
(vi) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;
(vii) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$;
From now on, we let L be a lattice implication algebra. Subset A of L is called an LI-ideal of L if it satisfies the following conditions:
(i) $0 \in A$;
(ii) $(\forall\, x, y \in L)\big((x \rightarrow y)' \epsilon A,\ y \epsilon A\ imply\ x \epsilon A\big)$;
LI-ideal A is called proper if $A \neq L$.
LI-ideal A of L is a prime LI-ideal if and only if $x \wedge y \in A$ implies $x \in A$ or $y \in A$, for any $x, y \in L$.
LI-ideal A of L is maximal, if it is proper and not a proper subset of any proper LI-ideal of L.
Subset A of L is an obstinate LI-ideal if $x \notin A$ and $y \notin A$ imply $(x \rightarrow y)' \in A$ and $(y \rightarrow x)' \in A$, for any $x, y \in L$.
Subset A of L is an ultra LI-ideal if it satisfies:
$x \in A \Leftrightarrow x' \notin A$
Theorem 3.2. ([16]) Let A be an LI-ideal of L. If $(x \rightarrow y)' \epsilon A, (y \rightarrow z)' \epsilon A$. Then $(x \rightarrow z)' \in A$.
Theorem 3.3. ([16]) Let A be a proper LI-ideal of L. A is a prime LI-ideal if and only if $(x \rightarrow y)' \in A$ or $(y \rightarrow x)' \in A$.
Theorem 3.4. ([16]) Let A be a proper LI-ideal of L. Then the following assertions are equivalent:
(i) A is a proper ultra LI-ideal.
(ii) A is a proper prime LI-ideal.

(iii) A is an obstinate LI-ideal.
Definition 3.5. ([16]) Let A be an LI-ideal of L and define relation $\equiv_A$ on L as follows:

$$x \equiv_A y \Leftrightarrow (x \to y)' \in A, (y \to x)' \in A.$$

It is easily verified that $\equiv_A$ is a congruence relation. Let $\frac{L}{A}$ be a set of congruence classes of $\equiv_A$. Therefore $\frac{L}{A} := \{[x]_A | x \epsilon L\}$, where $[x]_A := \{y \in L | x \equiv_A y\}$. Therefore, $\left(\frac{L}{A}, \to, [0]_A\right)$ is a lattice implication algebra, $[x]_A \to [y]_A = [x \to y]_A$.

4- Graphs of lattice implication algebras based on LI- ideal

In this section, we characterize LI-ideal based graphs $\Gamma^A(L)$ and $\Gamma_A(L), respectively$.

Definition 4.1. Let A be an LI-ideal of L. Then we have:
(i) $\Gamma^A(L) = (L, E^A)$ is a graph with vertices L and edges E^A, where $xy \epsilon E^A$ if and only if $(x \to y)' \epsilon A$ and $(y \to x)' \epsilon A$, for any $x, y \in L$.
(ii) $\Gamma_A(L) = (L, E_A)$ is a graph with vertices L and edges E_A, where $xy \epsilon E_A$ if and only if $(x \to y)' \epsilon A$ or $(y \to x)' \epsilon A$, for any $x, y \in L$.
Example 4.2. Let $L = \{0, a, b, c, d, I\}$ and the operation $\to$ is given by the table 1:

Table1- Implication operator

$\to$	0	a	b	c	d	I
0	I	I	I	I	I	I
a	c	I	b	c	b	I
b	d	a	I	b	a	I
c	a	a	I	I	a	I
d	b	I	I	b	I	I
I	0	a	b	c	d	I

Therefore, $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. It is easy to verify that $A = \{0, c\}$ is an LI-ideal of L. Also, we can see $\Gamma^A(L)$ is an empty graph, $E(\Gamma_A(L)) = \{0a, 0b, 0c, 0d, 0I, ad, bc, aI, bI, cI, dI\}$.
Theorem 4.3. Let A be an LI-ideal of L. Then $diam(\Gamma_A(L)) \leq 2, gr(\Gamma_A(L)) = 3$.
Proof. It is known that vertices $0, I$ connects to any element in L. Since $(0 \to x)' = 0 \epsilon A$ and $(x \to I)' = 0 \epsilon A$, $diam(\Gamma_A(L)) \leq 2, gr(\Gamma_A(L)) = 3$.
Theorem 4.4. Let A be an LI-ideal of L. Then
(i) $\Gamma_A(L)$ is regular if and only if it is complete.
(ii) If $\Gamma^A(L)$ is regular, then it is a complete graph on A.
(iii) If L be a chain, $\Gamma_A(L)$ is complete.
Proof. (i)($\Rightarrow$) Let $\Gamma_A(L)$ be a regular graph. Since $(0 \to x)' = 0 \epsilon A$ and $(x \to I)' = 0 \epsilon A$ for any $x \in L$, $\deg(0) = \deg(I) = |L| - 1$. Now, since $\Gamma_A(L)$ is regular for any $x \in L, \deg(x) = \deg(0) = \deg(I) = |L| - 1$ and so on for any $x \in L, \deg(x) = |L| - 1$. This means that $\Gamma_A(L)$ is a complete graph.
($\Leftarrow$) It is clear that any complete graph is regular.
(ii) Let $\Gamma^A(L)$ be a regular graph. Since $(0 \to x)' = 0 \in A$ and $(x \to 0)' = x \in A$ for any $x \in A, \deg(0) = |A| - 1$. Now, since $\Gamma^A(L)$ is regular for any $x \in A, \deg(x) = \deg(0) = |A| - 1$. Thus, $\Gamma^A(L)$ is complete on LI-ideal A.
(iii) If L is a chain, then for any $x, y \in L, x \leq y \; or \; y \leq x$ and so $(x \to y)' = 0 \in A \; or \; (y \to x)' = 0 \in A$. Therefore, $\deg(x) = |L| - 1$, for any $x \in L$ and $\Gamma_A(L)$ is complete.
Theorem 4.5. Let A be an ultra LI-ideal of L. Then, $\deg(I) = |L| - |A|$ in $\Gamma^A(L)$.

Proof. It is known that $(x \to I)' = 0 \in A$, since A is an ultra LI-ideal $(I \to x)' \in A$ for all $x \notin A$. Therefore, by Definition 4.1 of $\Gamma^A(L)$, $\deg(I) = |L| - |A|$.

Theorem 4.6. Let A be an LI-ideal of L. Then, 0 and I are not orthogonal in the graph $\Gamma_A(L)$ and $\Gamma_A(L)$ is not complemented.

Proof. According to Theorem 4.3 every vertex in the graph $\Gamma_A(L)$ connected to both 0 and I. Thus, 0 and I are not orthogonal. Moreover, there are not vertices x and y in $\Gamma_A(L)$ such that x and y are orthogonal. Therefore, $\Gamma_A(L)$ is not complemented.

Theorem 4.7. Let A be an LI-ideal of L. Then $S_1 = \{0\}$ and $S_2 = \{I\}$ are two dominating sets in graph $\Gamma_A(L)$. Therefore, $\gamma(\Gamma_A(L)) = 1$.

Proof. Straightforward by Definition 2.1 of dominating set and by Theorem 4.5.

Theorem 4.8. Let $A = \{0\}$ be an LI-ideal of L. Then $Com(L) = \Gamma_A(L)$, where $Com(L)$ is a comparability of L.

Proof. Let $x, y \in L, xy \in E\left(\Gamma_{\{0\}}(L)\right)$. Therefore, by Definition 4.1 of graph $\Gamma_A(L)$, $(x \to y)' = 0$ or $(y \to x)' = 0$ Thus, $x \le y$ or $y \le x$, and thus, $xy \in E(Com(L))$. The converse is clear.

Theorem 4.9. Let L be semimodular of rank d that is not a chain. Then $\Gamma_{\{0\}}(L)$ is $(d+1)$ $-$connected if and only if L has no simplicial element , where $z \in L$ is simplicial if the elements comparable to z form a chain.

Proof. By Theorems 2.14 and 4.8.

Proposition 4.10. Let A be an LI-ideal of L. Then $\omega(\Gamma_A(L)) \ge \max\{|B|; B \text{ } is \text{ } a \text{ } chain \text{ } in \text{ } L\}$.

Proof. Let B be a chain in L. Then for all $x, y \in B, x \le y \text{ } or \text{ } y \le x$. In other words, $(x \to y)' = 0 \in A$ or $(y \to x)' = 0 \in A$.

Thus, $xy \in E(\Gamma_A(L))$ by Definition 4.1 of graph $\Gamma_A(L)$, since $\omega(\Gamma_A(L))$ is the length of the greatest induced complete subgraph in the graph $\Gamma_A(L)$. Therefore, $\omega(\Gamma_A(L)) \ge \max\{ |B|; B \text{ } is \text{ } a \text{ } chain \text{ } in \text{ } L\}$.

Theorem 4.11. Let A be an LI-ideal of L. Then $\Gamma_A(L)$ is an Euler graph if and only if |L| is odd.

Proof. Theorem 4.3 says that $\Gamma_A(L)$ is connected. So, by Euler's theorem, $\Gamma_A(L)$ is an Euler graph if and only if the degree of any vertex is even. Therefore, if $\Gamma_A(L)$ is an Euler graph, then $deg(0)$ is even. On the other hand, by Theorem 4.2, deg(0) = |L|-1. Therefore, if $\Gamma_A(L)$ is an Euler graph, then $|L|$ is odd. Hence, this is proved.

Theorem 4.12. If $I \in A$, then the following statements hold:

(i) $\Gamma^A(L)$ is planar if and only if $|L| \le 4$.

(ii) $\Gamma^A(L)$ is outer-planar if and only if $|L| \le 3$.

(iii) $\Gamma^A(L)$ is toroidal if and only if $|L| \le 7$.

Proof. (i) If $I \in A$, then $A = L$. Hence, $\Gamma^A(L)$ is a complete graph, if $|L| > 4$ then $\Gamma^A(L)$ has an induced sub-graph isomorphic to K_5. So, by Kuratowski's theorem , $\Gamma^A(L)$is not planar.

(ii) If $I \in A$, then $A = L$. Hence, $\Gamma^A(L)$ is a complete graph, if $|L| > 3$ then $\Gamma^A(L)$ has an induced sub-graph isomorphic to K_4. So, by Definition 2.5 , $\Gamma^A(L)$is not outer-planar.

(iii) If $I \in A$, then $A = L$. Hence, $\Gamma^A(L)$ is a complete graph, if $|L| > 7$ then $\Gamma^A(L)$ has an induced sub-graph isomorphic to K_8. So, by Theorem 2.7 , $\Gamma^A(L)$is not toroidal.

Theorem 4.13. Let A be an LI-ideal of L. Then there is not $m, n \in N$ in such away that $\Gamma^A(L)$ be isomorphic to $K_{m,n}$.

Proof. Let there be $m, n \in N$ so that $\Gamma^A(L)$ is isomorphic to $K_{m,n}$. Then there are the sets $A = \{x_1, \dots, x_m\}$ and $B = \{y_1, \dots, y_n\}$ in such away that $(x_i \to y_j)' \in A$ and $(y_j \to x_i)' \in A$ for all $i = 1, \dots, m, j = 1, \dots, n$. By transitive property $\to$, $(x_i \to x_k)' \epsilon A$ and $(y_j \to y_l)' \in A$ for all $i, k \in \{1, \dots, m\}, j, l \in \{1, \dots, n\}$. This is a contradiction to $K_{m,n}$ being a complete bipartite graph. Thus, this is proved.

Proposition 4.14. Let $A = \{0, a\}$ be an LI-ideal of L, where $a \in atom(L), B = \{x \in L | x\ covers\ a\}$. Then we have:
(i) If $|B| \geq 3$, then $\Gamma_A(L)$ is not planar.
(ii) If $|B| \geq 2$, then $\Gamma_A(L)$ is not outer-planar.
(iii) If $|B| \geq 7$, then $\Gamma_A(L)$ is not toroidal.
Proof. (i) Since $|B| \geq 3$, the subset $B' = \{x_1, x_2, x_3\}$ of the set B can be chosen. It is clear that for all $i = 1, 2, 3, (0 \to x_i)' = 0, (a \to x_i)' = 0\ and\ (x_i \to I)' = 0$. So, the graph of $\Gamma_A(L)$ on $B' \cup \{0, a, I\}$ has a sub-graph isomorphic to $K_{3,3}$. Thus by Kuratowski's theorem, $\Gamma_A(L)$ is not planar.
(ii) Since $|B| \geq 2$, the subset $B' = \{x_1, x_2\}$ of the set B can be chosen. It is clear that for all $i = 1, 2, (0 \to x_i)' = 0, (a \to x_i)' = 0\ and\ (x_i \to I)' = 0$. So, the graph of $\Gamma_A(L)$ on $B' \cup \{0, a, I\}$ has a sub-graph isomorphic to $K_{2,3}$. Thus by Definition 2.5, $\Gamma_A(L)$ is not outer-planar.
(iii) Since $|B| \geq 7$, the subset $B' = \{x_1, \dots, x_7\}$ of the set B can be chosen. It is clear that for all $i = 1, \dots, 7, (0 \to x_i)' = 0, (a \to x_i)' = 0\ and\ (x_i \to I)' = 0$. So, the graph of $\Gamma_A(L)$ on $B' \cup \{0, a, I\}$ has a sub-graph isomorphic to $K_{3,7}$. Thus by theorem 2.7, $\Gamma_A(L)$ is not toroidal.
Theorem 4.15. Let A be an LI-ideal of L. Then $A = \{0\}$ if and only if graph $\Gamma^A(L)$ is empty that is $E^A = \emptyset$.
Proof. Let $xy \in E(\Gamma^A(L)), x, y \in L$. Then $(x \to y)' \in A = \{0\}$ and $(y \to x)' \in A = \{0\}$. So, $x = y$. Therefore, $\Gamma^A(L)$ is an empty graph. Conversely, let $\Gamma^A(L)$ be an empty graph. Therefore, if for all $x, y \in L, xy \in E(\Gamma^A(L))$, then $x = y$. In other words if for all $x, y \in L, (x \to y)' \in A$ and $(y \to x)' \in A$, then $x = y$. Thus, $(x \to y)' = (x \to x)' = 0 \in A$. Thus, A={0}, is proved.
Lemma 4.16. We have for any $x, y \in L, (y \to x)' = y$ if and only if $x \wedge y = 0$.
Theorem 4.17. Let $A = \{0, a\}$ be an LI-ideal of L and $B = \{x \in L | x\ covers\ a\}$, where $a \in atom\ (L)$. If $\Gamma_A(L)$ is planar, then one of the following statements holds:
(i) $|B| = 1$.
(ii) | B|=2, and $|U_x^y| \leq 2$ for all $x, y \in B$, and if $|U_x^y| = 2$, then $|U_y^x| = 1$, and if $|U_y^x| = 2$, then $|U_x^y| = 1$.
Proof. Since $\Gamma_A(L)$ is planar by Proposition 4.14, $|B| \leq 2$. Suppose that $|B| \neq 1$. Thus, $|B| = 2$. Set $B \coloneqq \{x, y\}$. If $|U_y^x| \geq 3$ or $|U_x^y| \geq 3$, then $\Gamma_A(L)$ has a sub-graph isomorphic to $K_{3,3}$. Without loss of generality, suppose $|U_y^x| \geq 3$. Thus, there exist $\{a_1, a_2, a_3\} \in U_y^x$ so denote $V_1 = \{0, a, y\}$ and $V_2 = \{x, a_1, a_2\}$, then we have $(0 \to x)' = 0, (0 \to a_i)' = 0, (a \to a_i)' = 0$, and $(a \to x)' = 0, i = 1,2$, since $x \wedge y, a_1 \wedge y, a_2 \wedge y = a \neq 0$, since $(y \to x)', (y \to a_1)'$, and $(y \to a_2)' \leq y$, by Lemma 4.16, we have $(y \to x) = a, (y \to a_1) = a, (y \to a_2) = a$ and so $\Gamma_A(L)$ is not planar. Which is impossible. Hence, $|U_y^x| \leq 2$ and $|U_x^y| \leq 2$. Now, suppose that $|U_x^y| = 2$ and $|U_y^x| = 2$. Set $V_1 = U_x^y \cup \{a\}$ and $V_2 = U_y^x \cup \{0\}$. It is easy to see that $\Gamma_A(L)$ has a sub-graph isomorphic to $K_{3,3}$ with parts V_1 and V_2 , which is impossible. So, if $|U_y^x| = 2$, then $|U_x^y| = 1$. Also, if $|U_y^x| = 2$, then $|U_x^y| = 1$.
Theorem 4.18. Let A be an LI-ideal of L, $B = \{x, y \in L;\ (x \to y)' \notin A, (y \to x)' \notin A\}$. Therefore, we have the following statements:
(i) $\Gamma_A(L - B)$ is planar if and only if $\Gamma_A(L)$ is planar.
(ii) $\Gamma_A(L - B)$ is outer- planar if and only if $\Gamma_A(L)$ is outer-planar.
(iii) $\Gamma_A(L - B)$ is toroidal if and only if $\Gamma_A(L)$ is toroidal.
Proof. Straightforward.
Theorem 4.19. Let A be an LI-ideal of L. Then we have:
(i) $\Gamma^A([x]_A)$ is a complete graph, for any $x \in L$
(ii) $\Gamma^A(L) = \bigcup_{x \in L} \Gamma^A([x]_A)$,

(iii) $\Gamma^A(L)$ is a graph with $|\frac{L}{A}|$ $components$,
(iv) $\Gamma^A(L)$ is a planar graph if and only if $|[x]_A| \leq 4$, for all $x \in L$,
(v) $\Gamma^A(L)$ is an outerplanar graph if and only if $|[x]_A| \leq 3$, for all $x \in L$,
(vi) $\Gamma^A(L)$ is a toroidal graph if and only if $|[x]_A| \leq 7$, for all $x \in L$,
(vii) $\omega(\Gamma^A(L)) = \max\{|[x]_A|; x \in L\}$.

Proof. (i) Let $u, v \in [x]_A$ then by Definition 3.5 of $\equiv_A$, $(u \to x)' \in A, (x \to u)' \in A, (v \to x)' \in A, and\ (x \to v)' \in A$ so $(u \to v)' \in A$ and $(v \to u)' \in A$ since Theorem 3.2 says operation $\to$ has transitive property, thus by Definition 4.1 of graph $\Gamma^A(L), uv \in E(\Gamma^A([x]_A))$ then $\Gamma^A([x]_A)$ is a complete graph.
(ii) Since $L = \bigcup_{x\in L}[x]_A$ then $V(\Gamma^A(L)) = V(\bigcup_{x \in L}\Gamma^A(([x]_A)))$. Clearly, $E(\bigcup_{x \in L}\Gamma^A(([x]_A))) \subseteq E(\Gamma^A([x]_A))$. Now, let $xy \in E(\Gamma^A(L))$. Then, $(x \to y)' \in A$ and $(y \to x)' \in A$, and so $xy \in \Gamma^A([x]_A)$. Hence, $E(\bigcup_{x \in L}\Gamma^A([x]_A)) = E(\Gamma^A(L))$.
(iii) We want to show that there is not any path between elements of $[x]_A$ and $[y]_A$ for all distinct elements $x, y \in L$. Let x, y be distinct elements of L, $a \in [x]_A$ and $b \in [y]_A$. If there is a path $a, a_1, a_2, \dots, a_n, b$ which link a to b, then $(a \to a_1)' \in A$ and so by Definition 3.5 of $\equiv_A$ we have $a_1 \in [a]_A = [x]_A$. By a similar way, we have $a_2, \dots, a_n, b \in [x]_A$ so $b \in [x]_A \cap [y]_A$ which is contrary to that $[x]_A \cap [y]_A = \emptyset$.
(iv) We know $\Gamma^A([x]_A)$ is a complete graph by (i), if $|[x]_A| > 4$ then $\Gamma^A(L)$ has induced subgraph isomorphic to K_5 , so by Kuratowski's Theorem $\Gamma^A([x]_A)$ is not planar.
(v) We know $\Gamma^A([x]_A)$ is a complete graph by (i), if $|[x]_A| > 3$ then $\Gamma^A(L)$ has induced subgraph isomorphic to K_4 , so by Definition 2.5, $\Gamma^A([x]_A)$ is not outerplanar.
(vi) We know $\Gamma^A([x]_A)$ is a complete graph by (i), if $|[x]_A| > 7$ then $\Gamma^A(L)$ has induced subgraph isomorphic to K_5 , so by Theorem 2.7, $\Gamma^A([x]_A)$ is not toroidal.
(vii) We know by (i) and (ii), $\Gamma^A([x]_A)$ is a complete graph, $\Gamma^A(L) = \bigcup \Gamma^A([x]_A)$, since $\omega(\Gamma^A(L))$ is length of greatest induced complete subgraph in the $\Gamma^A(L)$, we have $\omega(\Gamma^A(L)) = \max\{|[x]_A|; x \in L\}$.

Theorem 4.20. Let A be an LI-ideal of L. If $t = \left|\frac{L}{A}\right|$, $L = \bigcup_{i=1,\dots,t}[x_i]_A$. Then $\alpha(\Gamma^A(L)) \geq t$.

Proof. Let $z_1 \in [x_i]_A, z_2 \in [x_j]_A, i, j = 1, \dots, t$. By proof of Theorem 4.19, we have $z_1z_2 \notin E(\Gamma^A(L))$. Then by Definition 2.1 of an independent set that is a maximum size of the vertex set in such away that don't connect to each other, Theorem 4.19 (ii), we have $\alpha(\Gamma^A(L)) \geq t$.

Theorem 4.21. Let A be an LI-ideal of L. Then $\Gamma^A([x]_A)$ is an Euler graph if and only if $|[x]_A|$ is odd.

Proof. By Theorem 4.19, we know $\Gamma^A([x]_A)$ is a complete graph. If $|[x]_A|$ is dd then degree every vertex of $\Gamma^A([x]_A)$ is even, so by Theorem Euler that says a connected graph is an Euler graph if and only if degree every vertex is even, we gain $\Gamma^A([x]_A)$ is an Euler graph.

Theorem 4.22. Let L be finite, A be an LI-ideal of L. Then we have:

$$\chi(\Gamma^A(L), \lambda) = \prod_{a_t\in X}\chi(\Gamma^A([a_t]_A), \lambda) = \prod_{t=1,\dots,m}(\lambda - r_t + 1)(\lambda + 1)^{r_t}$$

Proof. Let $m \in N$, $\frac{L}{A} = \{[a_1]_A, \dots, [a_m]_A\}, [a_t]_A = \{x_{1,t}, \dots, x_{r_t,t}\}$ and $A_t = [b_{i,j}]$ be the adjacency matrix of $\Gamma^A([a_t]_A)$, for all $t \in \{1, 2, \dots, m\}$. Then $x_{1,1}, x_{2,1}, \dots, x_{r_1,1}, x_{1,2}, x_{2,2}, \dots, x_{r_2,2}, \dots, x_{1,m}, x_{2,m}, \dots, x_{r_m,m}\}$. Since $[a_i]_A \cap [a_j]_A = \emptyset$, for all distinct $i, j \in \{1, 2, \dots, m\}$, then by Theorem 4.19 (ii), the adjacency matrix of $\Gamma^A(L)$ is of the form

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{bmatrix}$$

Where A_t is isomorphic to adjacent matrix of a complete graph K_{r_t} on r_t vertices, for all $t \in \{1, 2, \dots, m\}$. By the properties of the determinate, we have,

$$\chi(\Gamma^A(L), \lambda) = \det(\lambda I - A) = \det(\lambda I_1 - A_1) \times \det(\lambda I_2 - A_2) \times \dots \times \det(\lambda I_m - A_m) = \prod_{a_t \in X} \chi(\Gamma^A([a_t]_A), \lambda)$$

, where I_t is a $r_t \times r_t$ identity matrix, for all $t \in \{1,2, \dots, m\}$. On the other hand by Theorem 4.19(i), we have

$$\chi(\Gamma^A(L), \lambda) = \prod_{t=1,\dots,m} (\lambda - r_t + 1)(\lambda + 1)^{r_t}$$

Theorem 4.23. Let L be finite and t be the number of element $a \in L$ such that $|[a]_A|$=1. Then, we have

(i) $\chi(\Gamma^A(L), \lambda) = \lambda^t \times f(\lambda)$, for some polynomial $f(\lambda)$.

(ii) $A = \{0\}$ if and only if $\chi(\Gamma^A(L), \lambda) = \lambda^n$, for some $n \in N$.

Proof. Let $|L| = n$ and $\{a_1, \dots, a_t\}$ be the set of all elements of L such that $|[a_i]_A| = 1$, for all $i \in \{1,2, \dots, t\}$. Then by using the proof of Theorem 4.19, the adjacent matrix of $\Gamma^A(L)$ is of the form

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}_{n \times n}$$

Where B is an $(n - t \times n - t)$ matrix. Hence, by properties of the determinate, we have $\chi\,(\Gamma^A(L), \lambda) = \det(\lambda I_t) \times det(\lambda I_{n-t} - B) = \lambda^t \times det(\lambda I_{n-t} - B)$. Let $f(\lambda) = det(\lambda I_{n-t} - B)$, then $\chi(\Gamma^A(L), \lambda) = \lambda^t \times f(\lambda)$.

(ii) Since $A = \{0\}$, then $|[a]_A| = 1$, for all $a \in L$. Therfore, by (i), $\chi(\Gamma^A(L), \lambda) = \lambda^n$, where $|L| = n$. Conversely, let $\chi(\Gamma^A(L), \lambda) = \lambda^n$, for some $n \in N$. Then $\Gamma^A(L)$ is an empty graph. Therefore by Theorem 4.15, $A = \{0\}$.

Theorem 4.24. Let A be an LI-ideal of L. Then $\Gamma^A(L)$ is chordal.

Proof. Let $x_0, x_1, \dots, x_n$ be a cycle of length $n \geq 4$. Then we have $(x_i \to x_{i+1})' \in A$ and $(x_{i+1} \to x_{i+2})' \in A$, for all $i = 0, \dots, n-2$.

By transitive property of $\to$, $(x_i \to x_{i+2})' \in A$ for all $i = 0, \dots, n-2$. Hence, $\Gamma^A(L)$ is chordal, complete proof.

Example 4.25. Consider $x_1, x_2, x_3, x_4 \in L$, such that $x_1 \leq x_2, x_1 \leq x_3, x_4 \leq x_2$ and $x_4 \leq x_3$, x_1 to x_4 and x_2 to x_3 are not comparable. Therefore, $\Gamma_A(L)$ has a cycle isomorphic to C_4 on the vertex set $\{x_1, x_2, x_3, x_4\}$. Since $(x_1 \to x_2)' = (x_1 \to x_3)' = (x_4 \to x_2)' = (x_4 \to x_3)' = 0 \in A$, by Definition 4.1 of graph $\Gamma_{\{0\}}(L), x_1x_2, x_1x_3, x_4x_2, x_4x_3 \in E\left(\Gamma_{\{0\}}(L)\right)$. On the other hand, $(x_2 \to x_3) \neq 0, (x_3 \to x_2) \neq 0, (x_1 \to x_4) \neq 0$, and $(x_4 \to x_1)' \neq 0$, since x_2 to x_3 and x_1 to x_4 is not comparable. Therefore, $x_1x_4, x_2x_3 \notin E\left(\Gamma_{\{0\}}(L)\right)$. Thus, graph $\Gamma_{\{0\}}(L)$ is not chordal. Hence, this is proved.

Theorem 4.26. Let A be an LI-ideal of L. Then, the following statements hold:

(i) If A is a prime LI-ideal. Then $\Gamma_A(L)$ is a complete graph.

(ii) If A is an ultra LI-ideal. Then $\Gamma_A(L)$ is a complete graph.

(iii) If A is an obstinate LI-ideal. Then $\Gamma_A(L)$ is a complete graph.

Proof. (i) Straightforward by Theorems 3.3, 3.4, Definition 4.1of graph $\Gamma_A(L)$.

(ii) Straightforward by Theorems 3.3, 3.4, Definition 4.1of graph $\Gamma_A(L)$.
(iii) Straightforward by Theorems 3.3, 3.4, Definition 4.1of graph $\Gamma_A(L)$.

5- Conclusions

In this paper, we introduce the zero divisor graphs $\Gamma^A(L)$ and $\Gamma_A(L)$ associated with lattice implication algebra regarding an LI-ideal A, where the vertex set of graphs $\Gamma^A(L)$ and $\Gamma_A(L)$ are the set of elements of L and two vertices x and y are adjacent in graph $\Gamma^A(L)$if and only if $(x \to y)' \in A$ and $(y \to x)' \in A$, and two distinct vertices x and y adjacent in graph $\Gamma_A(L)$ if and only if $(x \to y)' \in A$ or $(y \to x)' \in A$. In this article, we introduce concept of diameter, girth of graph. We show that $\Gamma^A(L)$ and $\Gamma_A(L)$ must be connected with diameter less than or equal 2, $gr(\Gamma_A(L)) = 3$.

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